Journal of Engineering Mathematics, Vol. 5, No. 4, October 1971
Wolters-Noordhoff Publishing-Groningen
Printed in the Netherlands

A Note on the Stochastic Theory of Time-Delayed Epidemics*

A. J. HERMANS

Technological University, Dept. of Mathematics, Delft, The Netherlands.

(Received January 20, 1971)

SUMMARY

Some deterministic and stochastic models for the spread of epidemics are studied. Some models are developed, which take into account a constant incubation time, but where the probability of a new infection is more general than in known models.

1. Introduction

In this paper we shall consider some models for the spread of epidemics. Both deterministic and stochastic models are studied. Early work in the mathematical theory of epidemics was mainly concerned with the development of deterministic models. It is obvious that if we assume a deterministic causal mechanism for the spread of a disease the number of infected people at some time will always be the same if the initial conditions are identical. Evidently from observations of epidemics follow that a number of random factors determine the development of an epidemic. Therefore it is preferable to use probabilistic or stochastic models to describe the phenomena.

An account of the theory of epidemics and an extensive bibliography are given by N. T. J. Bailey [1]. However, most of the models he describes do not take into account the effect of an incubation period. Bharucha-Reid [2] applies the Bellman-Harris theory [3] to a class of epidemic problems which take an incubation period into account. Bharucha-Reid considers the length of time, an individual is infected before infecting someone else as a random variable. However the chance of a new infection is independent on the seize of the infected population at each instant. In many epidemics this is not the realistic description. Therefore we develop some models which take into account a constant incubation time but where the probability of a new infection is more general.

2. Epidemic Without Removal

We first consider the total population consisting of members susceptible to infection and members that are infected. No members are removed from the population. Hence this is a rather unrealistic model because there is no hospitalization, death or recovery.

However, because of its simplicity this model serves us to illustrate the method and the results. As we mentioned before we consider an incubation period (defined as the length of time an individual is infected before infecting somenone else) to be a constant (τ sec).

We consider a population of $N+1$ individuals which are all susceptible to infection at time $t< -\tau$. At time $t=-\tau$ we infect one individual. Because the incubation period equals τ this individual can infect other individual at time $t=0$. So far the initial conditions. Now we shall describe the epidemic process as a deterministic one. We define:

$X(t)$ = number of infected individuals at time t, $N+1-X(t)$ = number of susceptible individuals at time t. We assume that the members of the population are homogeneously mixed. Therefore the rate of new infection is proportional to the number of infectives (individuals which are able to infect) and the number of susceptibles. We notice that the number of infectives

* The research for this paper was supported by the National Science Foundation under grant no. GP-18682 while on leave at the Courant Institute of Math. Sciences, New York University.

at time t equals $X(t-\tau)$. Therefore the number of infected individuals is determined by the differential equation

$$\frac{dX(t)}{dt} = \lambda X(t-\tau)\{N+1-X(t)\} \tag{2.1}$$

and the initial condition

$$X(t)=1 \quad \text{for} \quad -\tau \leqq t \leqq 0 \tag{2.2}$$

The constant λ may be absorbed in t by a proper time scale however, for reasons which will be obvious later, we choose $\lambda=(N+1)^{-1}$.

The value of $X(t)$ is easily calculated for finite values of τ and can be expressed in the form

$$X(t)=N+1-N \exp\left\{-(N+1)^{-1}\int_0^t X(s-\tau)ds\right\}. \tag{2.3}$$

We now consider the stochastic analogue of this problem. For this purpose we define some probability functions. Let $\Pr\{X(t)=i\}$ be the probability that the number of infected individuals equals i at time t. We now write down a basic law in probability theory to determine $\Pr\{X(t+h)=i\}$ as follows

$$\Pr\{X(t+h)=i\} = \sum_{k=0}^{i} \Pr\{X(t)=k\}\cdot\Pr\{X(t+h)=i/X(t)=k\} \tag{2.4}$$

and for a homogeneously mixed population we suppose that the conditional probability of a jump one is or order h, which leads in the case of no incubation period to:

$$\Pr\{X(t+h)=i\} = \left[1-\frac{i(N+1-i)}{N+1}h\right]\Pr\{X(t)=i\}+$$
$$+\frac{(i-1)(N+2-i)}{N+1}h\Pr\{X(t)=i-1\}+o(h)$$

We now use the notation $\Pr\{X(t)=i\}=P_i(t)$.

In this case we used the rule that the probability that a unit jump occurs during the time interval t, $t+h$ is proportional to the product of the number of infecting individuals at time t times the number of susceptible individuals at time t and proportional to h. A jump of size two and more occurs with a probability which is proportional to a higher power of h.

If we consider an epidemic with a finite incubation period we use the same rule with respect to the occurrence of a jump. However, the number of infecting individuals at time t equals the number of infected individuals at time $t-\tau$. We assume, the chance of a unit jump during the interval $t, t+h$ in the case j is given at time $t-\tau$ and i is given at time t equals

$$j(N+1-i)(N+1)^{-1}hP_j(t-\tau)$$

and therefore the total probability $P_i(t+h)$ can be written as:

$$P_i(t+h)=P_i(t)\sum_{j=0}^{N+1}\left\{1-\frac{j(N+1-i)}{N+1}h\right\}P_j(t-\tau)+$$
$$+P_{i-1}(t)\sum_{j=0}^{N+1}\frac{j(N+2-i)}{N+1}hP_j(t-\tau)+o(h) \tag{2.5}$$

If we define the expected value of $X(t)$ as $M(t)=E\{X(t)\}$, we arrive at the following equation if we let h go to zero

$$\frac{dP_i(t)}{dt}=\frac{M(t-\tau)}{N+1}\{-(N+1-i)P_i(t)+(N+2-i)P_{i-1}(t)\} \tag{2.6}$$

with initial condition

$$P_i(0)=\delta_i^1 \begin{cases} 1 & \text{for} \quad i=1 \\ 0 & \text{for} \quad i\neq 1 \end{cases}$$

At first sight it looks like we end up with one equation for 2 unknown functions $M(t)$ and $P_i(t)$ or a nonlinear equation in P_i alone. We get rid of the unknown $M(t)$ by solving the equation for tje expected value which we get by taking the expected value of (2.6)

$$\frac{dM(t)}{dt}=\frac{M(t-\tau)}{N+1}\{N+1-M(t)\}$$

with initial conditions.

$$M(t)=1 \quad \text{for} \quad -\tau\leqslant t\leqslant 0$$

Hence $M(t)$ is known and has the form (2.3). If we introduce a new variable σ as follows

$$\sigma=-\log[N^{-1}\{N+1-M(t)\}]=(N+1)^{-1}\int_0^t M(s-\tau)ds$$

equation (2.6) becomes

$$\frac{dP_i}{d\sigma}+(N+1-i)P_i=(N+2-i)P_{i-1}$$

with initial condition at $\sigma=0$, $P_i=\delta_i^1$ and the solution is

$$P_i(\sigma)=\binom{N}{i-1}e^{-N\sigma}(e^{\sigma}-1)^{i-1} \tag{2.7}$$

In the case of large populations i.e. N large we get the solution, by expansion of (2.7), of the form

$$P_i(\rho)=\frac{(\rho-1)^{i-1}}{(i-1)!}e^{-(\rho-1)} \tag{2.8}$$

where $\rho=M(t)$ is the solution of

$$\frac{dM}{dt}=M(t-\tau)$$

It can easily be shown that (2.8) is a solution of the equation

$$\frac{dP_i}{dt}=M(t-\tau)\{-P_i+P_{i-1}\} \tag{2.9}$$

which we get from (2.6) by taking $N+1\to\infty$. Although we may find the solution of (2.6) for large values of i in power series of $(N+1)^{-1}$ by expanding (2.7), it is worth while to develop an asymptotic approach of (2.6) directly by means of a method which is useful for more general stochastic problems.

3. Asymptotic Approach

We will use an asymptotic method which is similar to methods of geometrical optics and diffraction theory. We therefore introduce a small parameter $\varepsilon=(N+1)^{-1}$. Furthermore we define $u(x,t)=P_i(t)$ with $x=\varepsilon i$ and $m(t-\tau)=\tilde{m}(t)=\varepsilon M(t-\tau)$.

The equation for $u(x,t)$ has the form

$$\frac{\partial u}{\partial t}=\frac{\tilde{m}(t)}{\varepsilon}[-(1-x)u(x,t)+\{1-(x-\varepsilon)\}u(x-\varepsilon,t)] \tag{3.1}$$

In a paper which will be published, Ludwig [4] suggests the asymptotic solution being of the form

$$u(x,t)=e^{(1/\varepsilon)\Phi(x,t)}\{a(x,t)+\varepsilon a_1(x,t)+\dots\} \tag{3.2}$$

Substituting (3.2) in (3.1) and equating like powers of ε, we find

$$\Phi_t = m(t-\tau)(1-x)(e^{-\Phi_x}-1) \tag{3.3}$$

and

$$a_t = m(t-\tau)e^{-\Phi_x}\left[-(1-x)a_x + \left\{\frac{1-x}{2}\Phi_{xx}+1\right\}a\right] \tag{3.4}$$

The first equation may be called the eikonal equation and the second one the transport equation. As boundary condition for the eikonal equation we take

$$\Phi = 0 \text{ if } x=0 \quad \text{and} \quad \sigma = -\log(1-m(t)) = 0 .$$

The solution of (3.3) becomes

$$\Phi = -\sigma(1-x)-(1-x)\log(1-x)-x \log x + x \log(1-e^{-\sigma}) \tag{3.5}$$

and the solution of (3.4) equals

$$a(x,t) = K(1-x)^{-\frac{1}{2}}x^{-\frac{1}{2}}$$

where K is an arbitrary constant. Hence the asymptotic solution of (3.1) has the form

$$u(x,t) = K(1-x)^{-\{(1-x)/\varepsilon\}-\frac{1}{2}}x^{-(x/\varepsilon)-\frac{1}{2}}(1-e^{-\sigma})^{x/\varepsilon}e^{-(\sigma/\varepsilon)(1-x)} \tag{3.6}$$

To determine the constant K we have to match (3.6) with the solution for small values of i i.e. with the solution (2.8), then follows

$$K = (2\pi)^{-\frac{1}{2}}(N+1)^{-\frac{1}{2}}$$

and the asymptotic solution becomes

$$u(x,t) = (2\pi)^{-\frac{1}{2}}(N+1)^{-\frac{1}{2}}(1-x)^{-\{(1-x)/\varepsilon\}-\frac{1}{2}}\cdot x^{-(x/\varepsilon)-\frac{1}{2}}\cdot \\ \cdot(1-e^{-\sigma})^{(x/\varepsilon)}e^{-(\sigma/\varepsilon)(1-x)} \approx P_i(\sigma) \tag{3.7}$$

with

$$\sigma = -\log(1-m(t))$$

and

$$\frac{dm}{dt} = m(t-\tau)(1-m(t)), \quad m(t)=1 \quad \text{for} \quad -\tau \leqq t \leqq 0$$

As we mentioned before (3.7) also follows by expanding (2.7). However in many cases where no exact solution is available the asymptotic solution can be found in a similar way (see for instance Ludwig [4]).

4. Epidemic with Removal

As we mentioned the application of the epidemic model described in section 2 is limited to a rather non realistic case because no infected individuals have been removed by death, recovery or other reasons. Hence we expect a better description if we take this phenomena into account. We assume that a proportion of the infected population may be removed. It is obvious that here a delay may occur. The rate of removal may be proportional to the number of infected individuals at a time κ earlier. The case we treat is the more complicated, may be less realistic, case where this delay time κ equals zero. For arbitrary finite values of κ a similar model can be treated. At the same time a birth rate of new susceptible individuals may be taken into account. This gives no severe complications too.

The following quantities are defined:

$x(t)$ = number of susceptible individuals at time t,
$y(t)$ = number of infected individuals at time t,
$z(t)$ = number of removed individuals at time t.

The same incubation period τ will be considered.
Again we consider the infected and susceptible individuals to be mixed homogeneously. Hence, the differential equation describing the deterministic process may be written as follows

$$\frac{dx}{dt} = -\beta x(t)\,y(t-\tau)\,,$$

$$\frac{dy}{dt} = \beta x(t)\,y(t-\tau) - \rho y(t)\,, \tag{4.1}$$

$$\frac{dz}{dt} = \rho y(t)\,,$$

with initial conditions at $t=0$,

$x=N$, $z=0$ and at $-\tau \leqq t \leqq 0$, $y=1$

We normalize the time in such a way that $\beta=(N+1)^{-1}$ and ρ is a constant.

The solution of this set of equations is easily found to be:

$$x = N \exp\left\{-(N+1)^{-1}\int_0^t y(\sigma-\tau)\,d\sigma\right\},$$

$$y = \mathrm{e}^{-\rho t}\left[1 + \frac{N}{N+1}\int_0^t \mathrm{e}^{\rho s} y(s-\tau)\exp\left\{-(N+1)^{-1}\int_0^s y(\sigma-\tau)\,d\sigma\right\}ds\right], \tag{4.2}$$

$$z = N+1-x-y\,.$$

We define the probability function p_{ij} as follows: $p_{ij}(t)$=probability that $x(t)=i$ and $y(t)=j$ at time t.

With similar arguments that led to equation (2.6) we arrive at an equation for $p_{ij}(t)$ of the form:

$$\frac{dp_{ij}}{dt} = \frac{\tilde{M}(t)}{N+1}\{(i+1)p_{i+1,j-1} - ip_{ij}\} + \rho\{(j+1)p_{i,j+1} - jp_{ij}\} \tag{4.3}$$

with initial conditions

$$p_{ij}(0)=0\,, \qquad p_{N,1}(0)=1$$

In this equation $\tilde{M}(t)$ is the expected value of y at time $(t-\tau)$ hence

$$\tilde{M}(t) = E\{y(t-\tau)\}$$

The solution of (4.3) can be found by using the probability generating function

$$P(z,w,t) = \sum_{i,j} p_{ij} z^i w^j \tag{4.4}$$

To find an equation for P we multiply (4.3) with $z^i w^j$ and sum over all i, j.

This procedure leads to

$$\frac{\partial P}{\partial t} = \frac{\tilde{M}(t)}{N+1}(w-z)\frac{\partial P}{\partial z} + \rho(1-w)\frac{\partial P}{\partial w} \tag{4.5}$$

with initial condition $P(z,w;\,0) = z^N w$.

The solution of (4.5) is easily found to be

$$P = \left\{z + \int_0^t \tilde{m}(\sigma)[1+(w-1)\,\mathrm{e}^{-\rho(t-\sigma)}]\exp\left(-\int_0^\sigma \tilde{m}(\xi)\,d\xi\right)d\sigma\right\}^N \cdot$$

$$\cdot \exp\left(-N\int_0^t \tilde{m}(\xi)\,d\xi\right)\{1+(w-1)\,\mathrm{e}^{-\rho t}\} \tag{4.6}$$

where we use the notation $\tilde{m}(t) = \tilde{M}(t) \cdot (N+1)^{-1}$ where $\tilde{M}(t) = y(t-\tau)$ given by (4.2). From (4.6) we derive $p_{ij}(t)$ by means of the expansion of P in powers $z^i w^j$. This leads to the final result:

$$p_{ij}(t) = \frac{N!\exp\left(-N\int_0^t \tilde{m}(\xi)d\xi\right)}{(N-i-j)!\,i!\,(j-1)!}\left\{\int_0^t \tilde{m}(\sigma)\exp[-\rho(t-\sigma)]\exp\left(-\int_0^\sigma \tilde{m}(\xi)d\xi\right)d\sigma\right\}^{j-1}$$
$$\cdot\left\{\int_0^t \tilde{m}(\sigma)\{1-\exp[-\rho(t-\sigma)]\}\exp\left(-\int_0^\sigma \tilde{m}(\xi)d\xi\right)d\sigma\right\}^{N-i-j}\cdot$$
$$\cdot\ \frac{1-e^{-\rho t}}{j}\int_0^t \tilde{m}(\sigma)\exp[-\rho(t-\sigma)]\exp\left(-\int_0^\sigma \tilde{m}(\xi)d\xi\right)d\sigma +$$
$$+\frac{e^{-\rho t}}{N-i-j+1}\int_0^t \tilde{m}(\sigma)\{1-\exp[-\rho(t-\sigma)]\}\exp\left(-\int_0^\sigma \tilde{m}(\xi)d\xi\right)d\sigma \qquad (4.7)$$

From this solution several asymptotic expressions can be derived. For instance we may consider the case of large N, with i of the same order. Or we consider i,j both of the order N. All this can be done by substituting Stirling's formula in the faculties. An indirect asymptotic approach, as carried out in section 3, leads to the same answer.

REFERENCES:

[1] N. T. J. Bailey, *The mathematical theory of epidemics*, Charles Criffin and Company, Ltd. 1957.
[2] A. T. Bharucha-Reid, On the stochastic theory of epidemics, *Proceedings of the third Berkeley Symposium on mathematical statistics and probaility*, Vol. III, University of California press, Berkeley 1956.
[3] R. Bellman and T. E. Harris, On age-dependent binary branch processes, *Annals of Math.*, 55 (1952) 280–295.
[4] D. Ludwig: Asymptotic solution of stochastic epidemic equations, (to be published).

Journal of Engineering Math., Vol. 5 (1971) 289–294